Module 2-4
Review of Probability Theory

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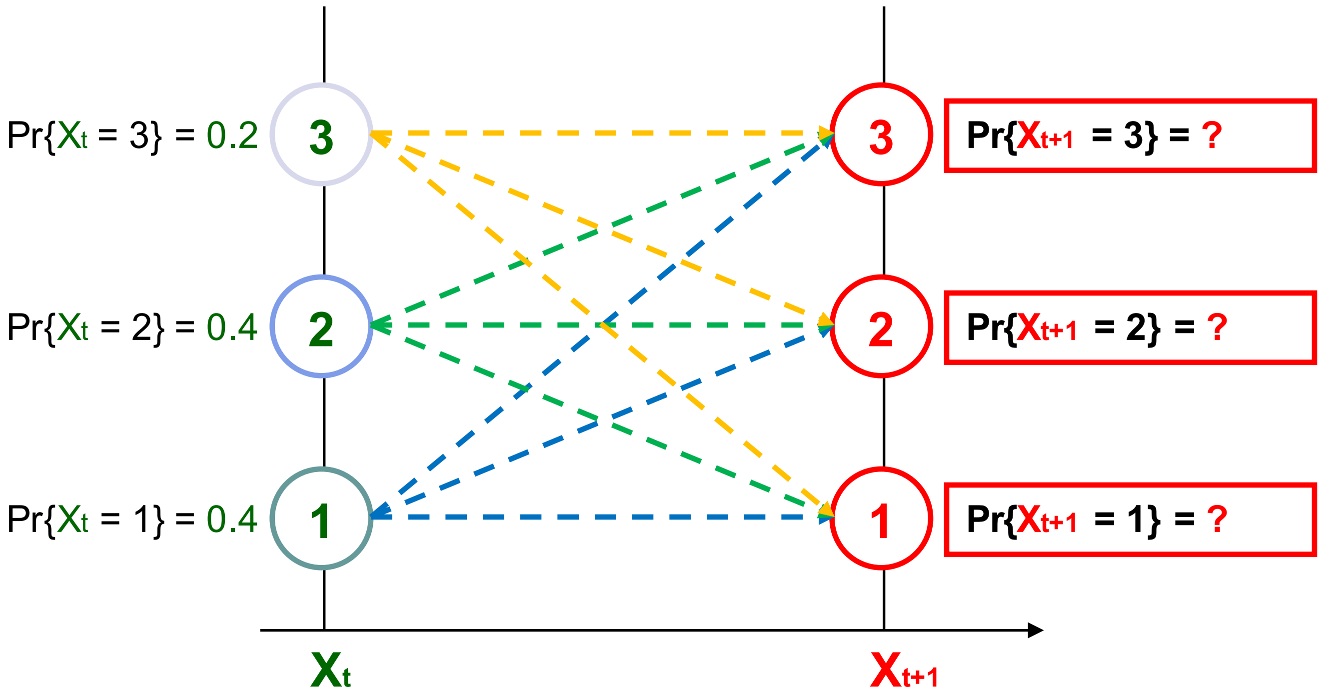
A Markov Process

The probability distribution of X_{t+1} is dependent only on the current state X_t.



Use concept of transition rate!

Transition Probability

Probability of being in state j at time t, given that the process started in state i, $P_{ij}(t)$

- Let X_{ij} be the random variable of time duration from state i to j, then

$$P_{ij}(\Delta t) = Pr\{\, t < X_{ij} < t + \Delta t \mid X_{ij} > t \,\}$$

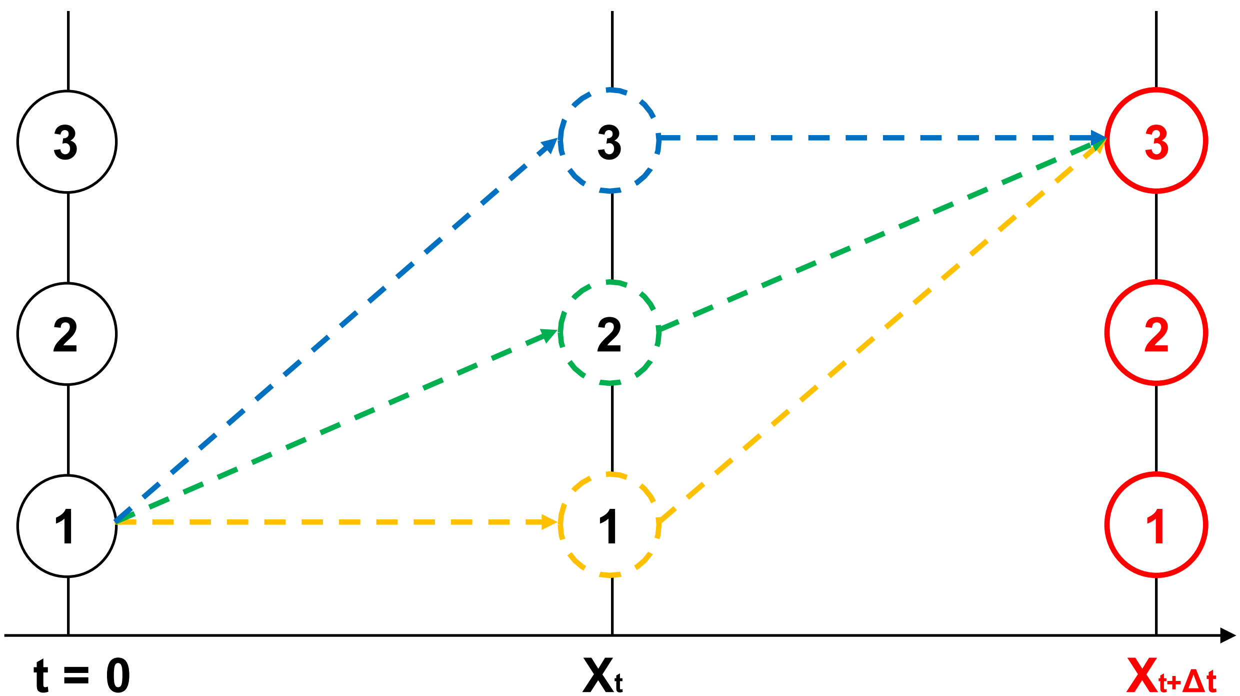
- From hazard rate as $\Delta t \rightarrow 0$,

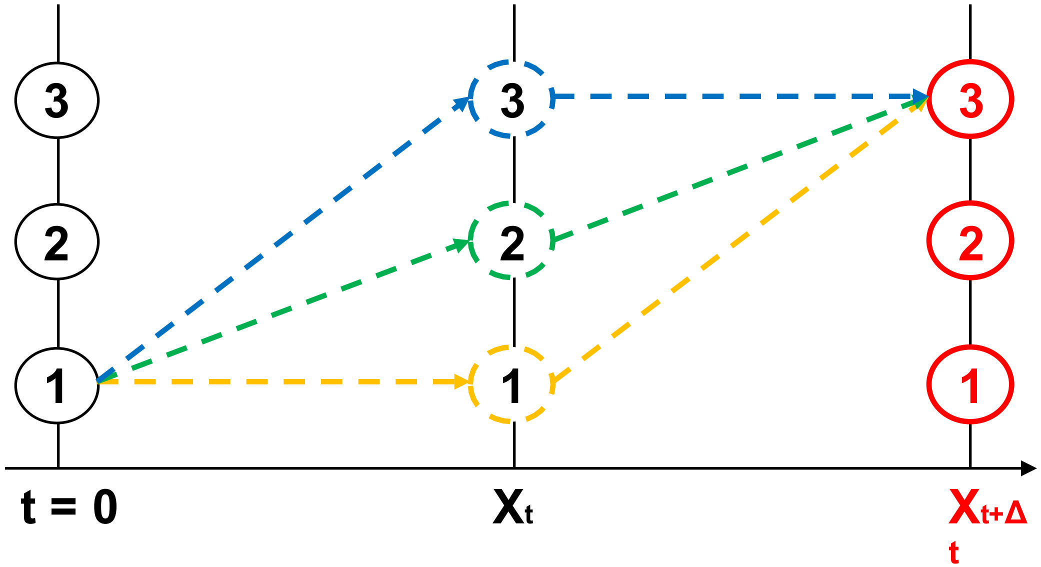
$$\lambda_{ij}\Delta t = Pr\{\, t < X_{ij} < t + \Delta t \mid X_{ij} > t \,\}$$

$$\boxed{P_{ij}(\Delta t) = \lambda_{ij}\Delta t}$$

A Three-State System

- Assume that the system started in state 1, find $P_{13}(t+\Delta t)$.





- From conditional probability as $\Delta t \rightarrow 0$,

$$P_{13}(t+\Delta t) = P_{11}(t)\, P_{13}(\Delta t) + P_{12}(t)\, P_{23}(\Delta t) + P_{13}(t)\, P_{33}(\Delta t)$$

$$= P_{11}(t)\, \lambda_{13}\, \Delta t + P_{12}(t)\, \lambda_{23}\, \Delta t + P_{13}(t)\, \lambda_{33}\, \Delta t$$

- Since $P_{31}(\Delta t) + P_{32}(\Delta t) + P_{33}(\Delta t) = 1$, then

$$\lambda_{33}\,\Delta t + \lambda_{32}\,\Delta t + \lambda_{31}\,\Delta t = 1$$

$$\lambda_{33}\,\Delta t = 1 - \lambda_{32}\,\Delta t - \lambda_{31}\,\Delta t$$

- We have,

$$P_{13}(t+\Delta t) = P_{11}(t)\lambda_{13}\,\Delta t + P_{12}(t)\,\lambda_{23}\,\Delta t + P_{13}(t)\,(1 - \lambda_{32}\,\Delta t - \lambda_{31}\,\Delta t)$$

- Then,

$$[P_{13}(t+\Delta t) - P_{13}(t)]/\,\Delta t = P_{11}(t)\,\lambda_{13} + P_{12}(t)\,\lambda_{23} + P_{13}(t)\,(-\,\lambda_{31} - \lambda_{32}) = P'_{13}(t)$$

- Thus,

$$P'_{13}(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t) \end{bmatrix} \begin{bmatrix} \lambda_{13} \\ \lambda_{23} \\ \lambda_{33} \end{bmatrix}$$

- Where $\lambda_{33} = -\left(\lambda_{31} + \lambda_{32}\right)$
- Similarly, we have

$$\begin{bmatrix} P'_{11}(t) & P'_{12}(t) & P'_{13}(t) \end{bmatrix} = \begin{bmatrix} P_{11}(t) & P_{12}(t) & P_{13}(t) \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

Transition Rate Matrix

$$R = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

- λ_{ij} is transition rate from state i to j, i ≠ j.
- where $\lambda_{ii} = -\Sigma\ \lambda_{ij}$

Steady State Probability

- From differential equation, at steady state the derivatives are zero,

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} R$$

- In addition, ΣP = 1,

$$P_1 + P_2 + P_3 = 1$$

Transient Behavior

$$P'(t) = P(t)R$$

As $\Delta t \to 0^+$

$$P(t+\Delta t) = P(t) + P'(t)\Delta t$$
$$= P(t)(I + R\,\Delta t)$$

It time t is divided into a very large number of equal intervals Δt, so that Δt is very small ($\doteq 0$), the above expression can be written as a recursive relationship

$$P(j\Delta t) = P(\overline{j-1}\;\Delta t)[I + R\,\Delta t]$$

Transient Behavior

$$P(j\Delta t) = P(\overline{j-1}\,\Delta t)[I + R\,\Delta t]$$

It should be noted that the above equation implies he approximation of a Markov process in continuous time by a discrete time Markov process with steps equal to Δt

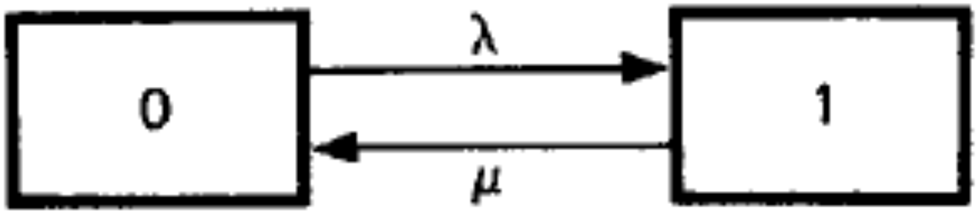
The (ij)th element of $(I + R\,\Delta t)$ is $\lambda_{ij}\,\Delta t$, i.e. the probability of transiting from state i to state j in one step of length Δt. Therefore $[I + R\Delta t]$ is a one step transition probability matrix. It can also be seen that

$$P(j\,\Delta t) = [I + R\,\Delta t]^j$$

which is the matrix multiplication technique in the discrete time case

Example

One of the processes commonly encountered in reliability studies is the two state Markov process. The state transition diagram for this process is shown below



Example

Transition rate matrix

$$R = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Take Laplace transform

$$P(s) = P(0)[sI - R]^{-1}$$

$$P(s) = P(0)\begin{bmatrix} s+\lambda & -\lambda \\ -\mu & s+\mu \end{bmatrix}^{-1}$$

$$= \frac{1}{s(s+\lambda+\mu)}\begin{bmatrix} s+\mu & \lambda \\ \mu & s+\lambda \end{bmatrix}$$

$$P(t) = \begin{bmatrix} \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \\ \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \end{bmatrix}$$

Example

The probability vector can be obtained

$$(p_0(t)p_1(t)) = (p_0(0)p_1(0))P(t)$$

$$p_0(t) = \frac{\mu}{\lambda+\mu}(p_0(0)+p_1(0)) + \left(p_0(0)\frac{\lambda}{\lambda+\mu} - p_1(0)\frac{\mu}{\lambda+\mu}\right)e^{-(\lambda+\mu)t}$$

$$= \frac{\mu}{\lambda+\mu} + \left(p_0(0) - \frac{\mu}{\lambda+\mu}\right)e^{-(\lambda+\mu)t}$$

and

$$p_1(t) = \frac{\lambda}{\lambda+\mu} + \left(p_1(0) - \frac{\lambda}{\lambda+\mu}\right)e^{-(\lambda+\mu)t}$$

As $t \to \infty$

$$\underset{t\to\infty}{p_0(t)} = p_0 = \frac{\mu}{\lambda+\mu} \qquad \underset{t\to\infty}{p_1(t)} = p_1 = \frac{\lambda}{\lambda+\mu}$$

Example

The steady state probability can also be obtained as

$$(p_0 \quad p_1)\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = 0$$

One of these identical equations can be used with

$$p_0 + p_1 = 1$$

to give

$$p_0 = \frac{\mu}{\lambda + \mu}$$

and

$$p_1 = \frac{\lambda}{\lambda + \mu}$$

First Passage Times

Denote the first passage time from state i to state j by T_{ij}, i.e. this is the time to enter state j for the first time starting in state i. If the state j is now made an absorbing state, the behaviour of the new stochastic process and the original process is the same until meeting j for the first time. If $p_{ij}(t)$ is the probability of being in state j, starting in state i for the new process then

$$P(T_{ij} \leqslant t) = p_{ij}(t)$$

The probability density function $f_{ij}(t)$ can be found by differentiation

$$f_{ij}(t) = \frac{d}{dt}(P(T_{ij} \leqslant t)) = \frac{d}{dt}p_{ij}(t)$$

The Laplace transform can be obtained by

$$\bar{f}_{ij}(s) = s\bar{p}_{ij}(s)$$

First Passage Times

The kth moment of the first passage time can be found by previously mentioned transform method

$$T_{ij}^{(k)} = (-1)^k \frac{d^k}{ds^k} \bar{f}_{ij}(s)\Big|_{s=0}$$

First Passage Times

If the absorbing state is the failed state, then the mean first passage time represents the MTTFF. The above procedure can be conveniently carried out in the matrix form. Let the states 1 to J be the elements of subset X^+ and J+1 to N be the elements of X^-. It is required to find the first passage time to the subset X^-. The matrix of transition rates can now be partitioned as follows

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

where

R_{11} is a $J \times J$ matrix

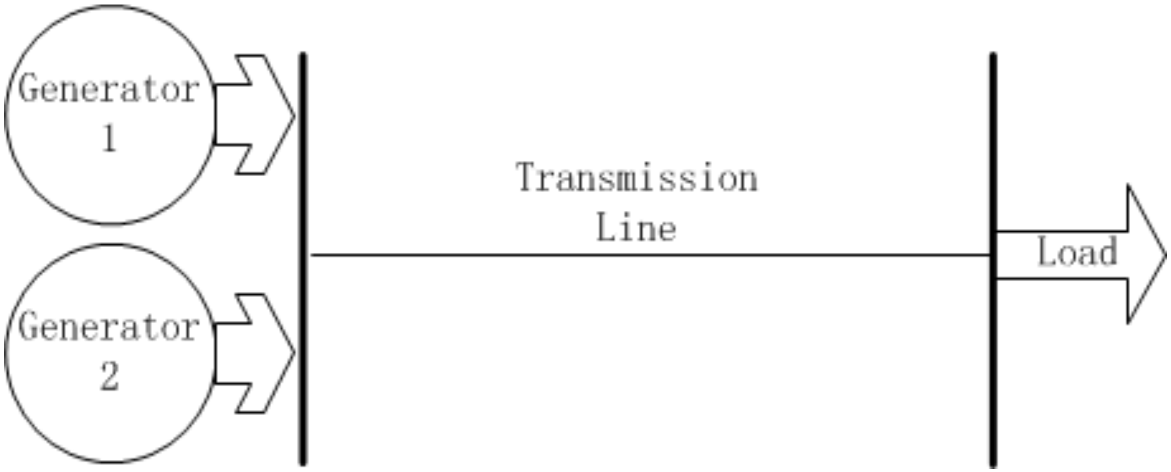
R_{12} is a $J \times (N-J)$ matrix

R_{21} is a $(N-J) \times J$ matrix

and

R_{22} is a $(N-J) \times (N-J)$ matrix

Example System

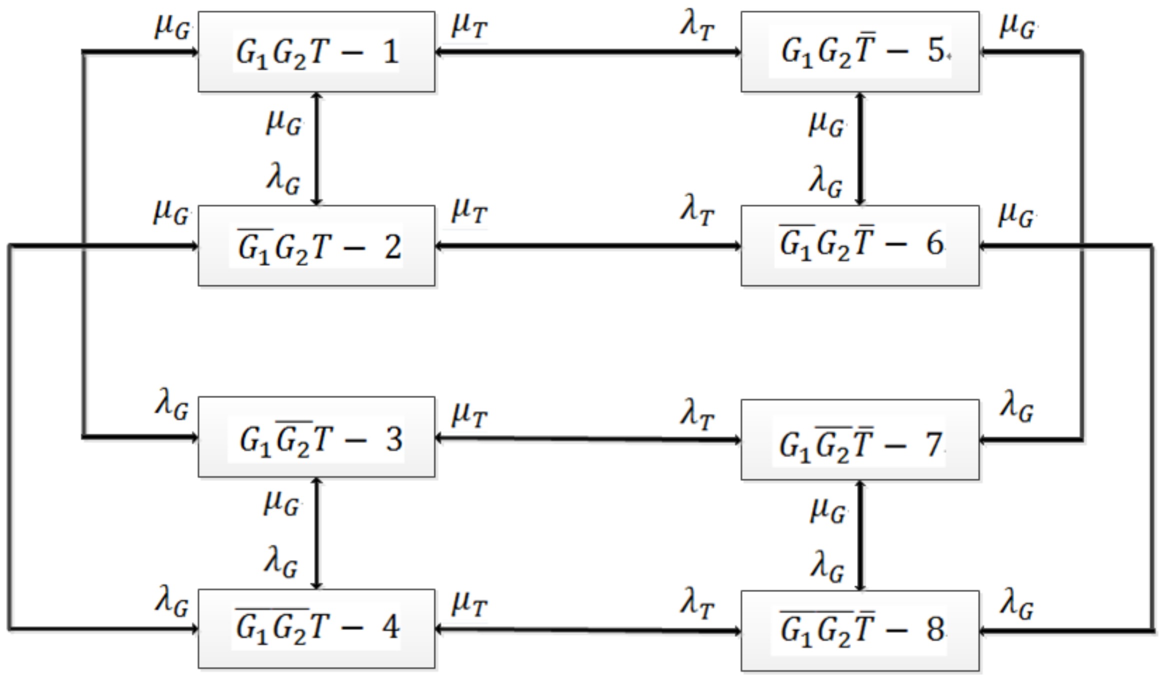


- Two generators are identical; each generator has generation capacity of 100MW, failure rate $\lambda\_G=0.1$/day, repair rate $\mu\_G=2$/day
- Transmission line's failure rate $\lambda\_T=0.01$/day, repair rate $\mu\_G=4$/day
- Load is 100MW

Define failure of this system as load being not supplied.

Assume all the failures of components are independent

Example System



State transition diagram

Example System

$$R = \begin{bmatrix} -(2\lambda_G+\lambda_T) & \lambda_G & \lambda_G & 0 & \lambda_T & 0 & 0 & 0 \\ \mu_G & -(\lambda_G+\lambda_T+\mu_G) & 0 & \lambda_G & 0 & \lambda_T & 0 & 0 \\ \mu_G & 0 & -(\lambda_G+\lambda_T+\mu_G) & \lambda_G & 0 & 0 & \lambda_T & 0 \\ 0 & \mu_G & \mu_G & -(\lambda_T+2\mu_G) & 0 & 0 & 0 & \lambda_T \\ \mu_T & 0 & 0 & 0 & -(2\lambda_G+\mu_T) & \lambda_G & \lambda_G & 0 \\ 0 & \mu_T & 0 & 0 & \mu_G & -(\lambda_G+\mu_G+\mu_T) & 0 & \lambda_G \\ 0 & 0 & \mu_T & 0 & \mu_G & 0 & -(\lambda_G+\mu_G+\mu_T) & \lambda_G \\ 0 & 0 & 0 & \mu_T & 0 & \mu_G & \mu_G & -(2\mu_G+\mu_T) \end{bmatrix}$$

Transition rate matrix

First Passage Times

The states $j \epsilon X^-$ are now absorbing states and therefore R_{21} and R_{22} are set to zero. Let $p(t)$ be the vector of state probabilities for an initial starting condition. This vector can be expressed as $(p_+(t)\, p_-(t))$ where $p_+(t)$ and $p_-(t)$ are the vectors containing the states $i \epsilon X^+$ and $i \epsilon X^-$ respectively. The forward differential equation now becomes

$$\frac{d}{dt}(p_+(t)p_-(t)) = (p_+(t)p_-(t)) \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}$$

Taking the Laplace transforms

$$s\bar{p}_+(s) - p_+(0) = \bar{p}_+(s)R_{11}$$

$$s\bar{p}_-(s) = \bar{p}_+(s)R_{12}$$

$p_-(0) = 0$ as the process started in $i \in X^+$. These equations can be rearranged as

$$\bar{p}_+(s) = p_+(0)[sI - R_{11}]^{-1}$$

and

$$s\bar{p}_-(s) = p_+(0)(sI - R_{11})^{-1}R_{12}$$

First Passage Times

The probability of being in subset X^- at time t is $p_-(t)U_{N-J}$

Where U_{N-J} is a unit column vector of dimension N-J

Previously, we have obtained

$$\bar{f}_{ij}(s) = s\bar{p}_{ij}(s)$$

Thus the Laplace transform of the probability density function of the first passage time is

$$\bar{f}\,(s) = sp_-(s)U_{N-J} = p_+(0)(sI - R)^{-1}R_{12}U_{N-J}$$

Since the rows of the transition rate matrix sum to zero

$$R_{12}U_{N-J} + R_{11}U_J = 0$$

Therefore

$$\bar{f}\,(s) = p_+(0)(sI - R_{11})^{-1}(-R_{11})U_J$$

First Passage Times

$$\bar{f}\,(s) = p_+(0)(sI - R_{11})^{-1}(-R_{11})U_J$$

$$T_{ij}^{(k)} = (-1)^k \frac{d^k}{ds^k} \bar{f}_{ij}(s)\Big|_{s=0}$$

The rth initial moment can be found

$$T^{(k)} = k!\, p_+(0)(-R_{11})^{-k}U_J$$

The mean is

$$\bar{T} = T^{(1)} = p_+(0)(-R_{11})^{-1}U_J$$

If the process started in the first state

$$\bar{T} = T^{(1)} = (1\ 0\ 0\ \ldots\ \ 0)(-R_{11})^{-1}U_J$$

First Passage Times

If X^{-} represents the failed condition, then $\bar{T}$ is the MTTFF. It should be realized that $\bar{T} = T^{(1)} = p_{+}(0)(-R_{11})^{-1}U_J$ can be derived from the theory of discrete time Markov chains by assuming that each step of the chain is $\Delta t \approx 0$.

The matrix of one step transition probability becomes $[I + R\Delta t]$ and by truncating the absorbing state, $\mathrm{Q} = [I + R_{11}\Delta t]$ and therefor the fundamental matrix

$$N = [I - Q]^{-1} = \frac{1}{\Delta t}[-R_{11}]^{-1}$$

This matrix gives the number of steps spent in the different states. The time spent in the different states can be obtained by multiplying by Δt i.e. the step length. From this point on it is easy to see that

$$\bar{T} = T^{(1)} = p_{+}(0)(-R_{11})^{-1}U_J$$

First Passage Times

The first passage time represents the time of entering a state or a set of states for the first time, starting in a particular state. It is sometimes necessary, however, to find the mean time spent in subset X^+ or X^-. For example, if X^+ and X^- represent the up and down states respectively, these time parameters represent the mean up time and the mean down time. In order to calculate these quantities, it is necessary to know the probabilities of beginning X^+ in the various states, which are its elements. Denoting the steady state probabilities of being in various states of the original process by p_i, the probability of beginning X^+ in state j is

$$p_j(0) = \frac{\sum_{i \in X^-} p_i \lambda_{ij}}{\sum_{j \in X^+} \sum_{i \in X^-} p_i \lambda_{ij}}$$

Here

λ_{ij} is the transition rate from state I to state j

In vector form

$$p_+(0) = \frac{p_- R_{21}}{p_- R_{21} U_k}$$

First Passage Times

In the steady state

$$p_+R_{11} + p_-R_{21} = 0$$

Therefore

$$p_+(0) = \frac{-p_+R_{11}}{p_+R_{11}U_k} = \frac{-p_+R_{11}}{p_+R_{12}U_{N-k}}$$

The mean stay in X^+ is

$$T^+ = \frac{-p_+R_{11}(-R_{11})^{-1}U_k}{p_+R_{12}U_{N-k}} = \frac{p_+U_k}{p_+R_{12}U_{N-k}}$$

$$= \sum_{i\in X^+} p_i \Big/ \sum_{i\in X^+} p_i \sum_{j\in X^-} \lambda_{ij}$$

$$= \sum_{i\in X^+} p_i \Big/ \sum_{i\in X^-} p_i \sum_{j\in X^+} \lambda_{ij}$$

First Passage Times

In a similar manner the mean duration in state X^-

$$T^- = \sum_{i \in X^-} p_i \Bigg/ \sum_{i \in X^-} p_i \sum_{j \in X^+} \lambda_{ij}$$

$$= \sum_{i \in X^-} p_i \Bigg/ \sum_{i \in X^+} p_i \sum_{j \in X^-} \lambda_{ij}$$

First Passage Times

The mean cycle time, i.e. the time between two successive encounters of X^+ or X^-

$$T = T^+ + T^-$$

$$= 1 \Big/ \sum_{i \in X^-} p_i \sum_{j \in X^+} \lambda_{ij}$$

$$= 1 \Big/ \sum_{i \in X^+} p_i \sum_{j \in X^-} \lambda_{ij}$$

These relationships will be derived from the frequency balancing technique in following lectures