CHAPTER 5

Techniques for Large Systems

Introduction

Several mathematical modelling techniques suitable for reliability evaluation have been illustrated. Although the problem of system reliability may appear conceptually simple, at least when constant transition rates can be assumed, the task may become difficult with large and complex systems. In control systems, communication networks and power networks, it is sometimes easy to derive the reliability block diagrams from the schematic diagrams. When the network approach can be used, the problem becomes relatively simple. In certain cases, however, it is difficult and often impossible to apply this approach. This is especially so with systems involving dependent failure or repair modes and those involving graded modes of operation. In such cases the state space approach is often the only method available. This chapter will identify problem areas while using this approach and suggest suitable techniques for overcoming these difficulties.

The Problem Areas

It was pointed out in the last chapter that the state space approach essentially involves the following steps:

1 *Evolution of the state space and the interstate transition rates.*

In a small system, it is possible to draw the state transition diagram of the system and then solve it with a calculator or program it for the digital computer. When, however, the state space is large, this procedure becomes impractical and often impossible. In such cases this process can be performed by a computer. The method was described in the last chapter. The basic idea is to let the states sequentially evolve by the realization of each possible transition mode of the components. When the transition modes of the components are dependent, system states may impose restrictions on some transition modes but in systems consisting of independent components, each component is allowed to realize all of its transition modes. In certain systems using the symmetry of the transition diagrams, special methods may be evolved for generating the state space and the interstate transition rates. The main problem in this step is to ensure that the correct state space and interstate transition rates have been generated. The generation of a correct state transition rate matrix is very important as this is the foundation for further calculations.

In some cases, the program may be used to generate the transition rate matrix of a similar but smaller system. The matrix along with the description of states can be printed out and checked visually. For ease in checking, fictitious transition rates, usually whole numbers may be used. When this is not possible or convenient, the program should have an independent checking subroutine. Each system state should be examined for the possible modes of transition. For each mode the resulting state description should be constructed and compared with the ones which have already been generated. If both the state description and the interstate transition rate agree, for all the states, it is reasonable to assume that the state transition rate matrix is correct. If at any state there is any discrepancy, the state and the resulting states should be printed. This will help in debugging.

The other problem is the size of the state space. As an example, a system consisting of n independent two state units will have 2^n possible states. The state transition matrix will be of $2^n \times 2^n$ size. The available memory in the computer may soon be exhausted. This difficulty can be alleviated to some extent by using the principles of sparsity programming. This problem will, however, be discussed in more detail later.

2 *Calculation of the state probabilities*

When the components are independent, the problem becomes simpler, since the system state probabilities can be derived from the component state probabilities by the simple multiplication rule of probabilities. Since the calculation of the probability of each system state can be done independently of the other states, evaluation can be made selectively for the states required for final calculation of the reliability measures. In the case of dependent transition modes the probabilities of all the states have to be calculated since the states can be solved independently for probabilities. If time dependent solutions are required, the state space equation has to be solved. The methods of doing it have already been discussed. Most often, however, the steady state solution is required. The problem is then reduced to solving a set of simultaneous linear equations. This is usually done using the Gauss elimination or Gauss–Jordan method. These techniques are explained in Appendix I. Though conceptually simple, the problem can become formidable even on the computer as the number of components becomes large, and the state space becomes very large. Computer storage limitations, the errors introduced by rounding off and the computation time required all make the problem a very difficult one. It should be appreciated that the transition rate matrix contains failure rates which are usually very small as compared to the repair rates which are comparatively large. The operations on small and large numbers are bound to introduce rounding off errors. It can therefore be appreciated that steps have to be taken to limit the state space.

3 *Calculation of the reliability measures*

The two key indices are the probability and frequency of encountering a certain configuration of states. The other indices, the mean cycle time and the mean down time can be then simply derived. Success or failure may not provide a complete description and it may be necessary to evaluate reliability measures for graded modes of operation e.g. in a power system it may be necessary to calculate the probabilities and frequencies of having different magnitudes of capacity deficiencies. Similarly in a transit system it may be necessary to know the probabilities and frequencies of having various numbers of vehicles available for transportation. The various states are then grouped into subsets denoting a particular condition. The probabilities and frequencies of these subsets are then calculated using the equations given in Chapter 3. In general all the states have to be scanned to classify them into the required subsets and to select the appropriate transition rates for frequency calculations. In many cases, the classification can be done by taking the advantage of the systematic pattern of the state space. But if all the states have to be scanned, it could be a time-consuming process.

It can be seen from the above discussion that the problem with large systems is essentially the size of the state space. In some situations the size of the state space does not grow proportionately with the number of components. The growth of the state space in such cases is restricted because of the dependency considerations. A familiar example is that of a series system of n components when the assumption is made that the exposure of components to failure is zero when the system is down. In this case the number of states is simply $n+1$ as compared with 2^n when the component failures are independent.

In general, the following procedure should be adopted.

1. The system should be divided into suitable subsystems which can be handled conveniently one at a time. A system is usually naturally subdivided into subsystems on the layout or functional basis. Most often this natural subdivision can be used as the basis for classification for reliability evaluation, but it is not necessary to do so. Primary concern is on the system effect of component failures. Every attempt should be made to divide the system into independent subsystems. The advantage in doing so is that the probabilities of the system can then be found by simple multiplication of the probabilities of the states of the subsystems. Another advantage is that the combination of the independent subsystems is simpler and the equivalent transition rate concept can be more conveniently employed. This will become clearer when the implications of the equivalent transition rate are considered. The independence of various subsystems may sometimes be achieved by shifting some components from one natural subsystem to another. This trick can sometimes prove quite useful.
2. The state space of each subsystem may be reduced either by merging states

or by truncating very low probability states. The principles of both of these techniques are explained in detail in this chapter.
3. The subsystems should then be combined into a complete system and the required reliability measures evaluated.

Two important concepts in large systems are therefore the merging of the states and truncation of states.

Equivalent Transition Rate and the Conditions of Mergeability

The equivalent transition rate concept was introduced in Chapter 3 and used for independent subsystems in Chapter 4, while using the network reduction approach. The principal use of the equivalent transition rate is in reducing the system or subsystem state space. The basic idea is to find a state space which is equivalent to the original state space but is more convenient to use. Assume that the entire state space is partitioned into m subsets

$X_i, i = 1, 2, \ldots, m$. The equivalent transition rate from subset X_p to X_q is obtained by using Equation (3.37)

$$\lambda_{pq}^{(e)}(t) = \sum_{i \in X_p} \sum_{j \in X_q} p_i(t)\lambda_{ij} \Big/ \sum_{i \in X_p} p_i(t) \tag{5.1}$$

This section examines the conditions under which this equivalent model will give the same results as would be obtained by using the original model. Complete knowledge of these conditions is quite important and the lack of awareness in this area can lead to gross errors. The concept of equivalent transition rate will be examined both in the transient domain and under the equilibrium conditions. This concept can be useful in the following ways:

(a) *System State Space Reduction*

Sometimes it may be desirable to reduce the state space of the system by merging together certain sets of states. The equivalent transition rates among the subsets of the lumped states are required.

State merging in the system state space is generally of value when the equivalent transition rates can be calculated without having to solve for the system state probabilities.

(b) *Subsystem State Space Reduction*

The most practical method is to break down the system into subsystems which can be individually solved and then to combine these solutions to get the results for the entire system. It may be desirable to reduce the state space of an individual subsystem and therefore the equivalent transfer rate will be calculated.

Model reduction in this case is useful even if all of the subsystem state probabilities have to be calculated to determine the equivalent transfer rates.

Transient Domain

The equivalent transfer rate is first examined from the point of view of system state space reduction. The equivalent transition rate from subset X_p to X_q is given by Equation (5.1).

The following observations can be made from the relationship in (5.1):

1. Since the state probabilities in Equation (5.1) depend upon the initial state probability vector, the equivalent transfer rate, $\lambda_{pq}^{(e)}(t)$ is a function of the initial state probability vector. In contrast to this the interstate transition rates λ_{ij} of the original state space are independent of any such condition. If $\lambda_{pq}^{(e)}(t)$ are to be independent of such a restriction they must then be independent of the state probabilities. This can happen if $\Sigma\lambda_{ij}$ is the same for all $i \epsilon X_p$ so that $j \epsilon X_q$

$$\lambda_{pq}^{(e)}(t) = \sum_{j \in X_q} \lambda_{ij} \sum_{i \in X_p} p_i(t) \Big/ \sum_{i \in X_p} p_i(t)$$

$$= \sum_{j \in X_q} \lambda_{ij} = \lambda_{pq}^{(e)} \tag{5.2}$$

2. Unless the expression for equivalent transfer rate is independent of the state probabilities as in Equation (5.2), $\lambda_{pq}^{(e)}(t)$ is obviously a function of t. If an explicit expression for $\lambda_{pq}^{(e)}(t)$ can be obtained, the solution for the reduced state space can be obtained by using Kolmogorov differential equations for the time specific transition rate matrix

$$A(u)P(t,u) = \frac{\partial P(t,u)}{\partial u} \tag{5.3}$$

where for $u > t$

$P(t,u)$ = The column matrix whose ith value $p_i(t,u)$ represents the probability of being in state i at time u for the given initial condition at t, i.e.

$$p_i(t,u) = P\{Z(u) = i \mid Z(t) = j\}$$

and $A(u)$ = The transpose of the time specific transition rate matrix. When the initial conditions are specified at the time of origin

$$A(u)P(0,u) = \frac{\partial P(0,u)}{\partial u} \tag{5.4}$$

When Equation (5.4) is used for the reduced system, the initial condition, $Z(0)$ should be the same as for the $\lambda_{pq}^{(e)}(u)$ in (5.1)

Writing the differential equation for the lumped state q

$$\frac{\partial p_q(0,u)}{\partial u} = \sum_{p \in X_r^-} \lambda_{pq}^{(e)}(u) p_p(0,u) - p_q(0,u) \sum_{p \in X_r^-} \lambda_{qp}^{(e)}(u) \tag{5.5}$$

where $X_r^- = \{\text{All states except } q\}$, the subscript r indicating that the reference is to the reduced state space.

Since the initial condition for (5.1) and (5.4) is the same

$$p_p(0,u) = \sum_{i \in X_p} p_i(u)$$

and

$$\frac{\partial p_q(0,u)}{\partial u} = \sum_{j \in X_q} p_j'(u)$$

The indices p and q refer to the lumped states whereas i and j indicate the original states. Substituting these values into Equation (5.5) and substituting the values of $\lambda_{pq}^{(e)}(t)$ from Expression (5.1)

$$\sum_{j \in X_q} p_j'(u) = \sum_{p \in X_r^-} \sum_{i \in X_p} \sum_{j \in X_q} p_i(u)\lambda_{ij} - \sum_{p \in X_r^-} \sum_{j \in X_q} \sum_{i \in X_p} p_j(u)\lambda_{ji}$$

Now if $X^+ = X_q$ and X^- is the disjoint subset, then the above equation can be written as

$$\sum_{j \in X^+} p_j'(u) = \sum_{i \in X^-} \sum_{j \in X^+} p_i(u)\lambda_{ij} - \sum_{j \in X^+} \sum_{i \in X^-} p_j(u)\lambda_{ji}$$

It can be seen that this is the same expression as would be obtained by following the argument used to derive Equation (3.7). The time specific equivalent transfer rates, therefore, do represent the process accurately but the essential point is that this does not solve anything because all the state probabilities have to be found before the equivalent transfer rates can be determined. The only case when the state probabilities do not have to be evaluated is the one given by Equation (5.2). In summary, mergeability from a systems state space viewpoint can be defined as follows:

Definition: If the entire state space is partitioned into m subsets X_i, $i = 1,2,...,m$ and the equivalent transfer rate from subset X_p to X_q, given by Equation (5.1), is time invariant and independent of the initial state probability vector, then the state space is said to be mergeable into the said partition.

The necessary and sufficient condition is that the transition rate from each state in subset X_p to each of the states in X_q when summed over all the states in X_q is the same for each state in X_p and the required equivalent rate is given by

$$\lambda_{pq}^{(e)} = \sum_{j \in X_q} \lambda_{ij}$$

When examining mergeability from the point of view of subsystem reduction, the state space of a subsystem S_a is assumed to be partitioned into subsets X_1^a, X_2^a, ..., X_p^a in such a manner that no information about the reliability analysis is lost. The equivalent transfer rate from one subset to another can be obtained using Equation (5.1). The important consideration for the present purpose, however, is that this equivalent rate should hold when this subsystem is combined with another subsystem. The equivalent transfer rate from the lumped states of subset X_l^a to those of X_m^a is given by

$$\lambda_{lm}^{a}(t) = \left(\sum_{i \in X_l^a} p_i(t) \sum_{j \in X_m^a} \lambda_{ij} \right) \Big/ \sum_{i \in X_l^a} p_i(t)$$

After merging together the states in each subset, there will be p equivalent states in the subsystem S_a, one equivalent state corresponding to one subset. This subsystem is now combined with another subsystem S_b with equivalent states $b_1, b_2, ..., b_q$. It is assumed that in the combined system there will be $p \times q$ states, there being a whole set of equivalent states a_u, $u = 1,..., p$, corresponding to every equivalent state in subsystem b. This can be indicated as follows

$$\begin{array}{llll} (b_1)a_1, & (b_1)a_2, & \ldots, & (b_1)a_p \\ (b_2)a_1, & (b_2)a_2, & \ldots, & (b_2)a_p \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (b_q)a_1, & (b_q)a_2, & \ldots, & (b_q)a_p \end{array}$$

It should be appreciated that this arrangement may not always be possible as some combinations may be incompatible. In the subsystems exposed to a common two state fluctuating environment, which may alter the component failure and repair rates, such an arrangement will exist in either state. Only the arrangement shown above will be discussed but the results will also hold for other cases. After combination of the two subsystems, certain interstate transition rates may be altered but it is assumed that the interstate transition rates with respect to the merged states are not changed so that the act of combining the subsystems does not affect the information contained in the merged states.

For the transfer rates to hold after combining, the equivalent transfer rate from $(b_u)a_l$ to $(b_u)a_m$ should be the same as that from a_l to a_m. Here u may be any number from 1 to q. Therefore, for mergeability

$$\lambda_{lm}^{ab}(t) = \lambda_{lm}^{a}(t)$$

i.e.,

$$\frac{\sum_{i \in X_l^a} p_i^u(t) \sum_{j \in X_m^a} \lambda_{ij}}{\sum_{i \in X_l^a} p_i^u(t)} = \frac{\sum_{i \in X_l^a} p_i(t) \sum_{j \in X_m^a} \lambda_{ij}}{\sum_{i \in X_l^a} p_i(t)} \tag{5.6}$$

where

$p_i^u(t)$ = The probability, for the given initial state, of being in state $i \in X_l^a$ corresponding to the condition represented by b_u in subsystem S_b.

The following conclusions can be drawn from the Equality (5.6):

(1) For independent subsystems

$$p_i^u(t) = p_i(t) p_{bu}(t)$$

where

$p_{bu}(t)$ = The probability of being in the equivalent state b_u for the given initial condition

Therefore

$$\begin{aligned} \lambda_{lm}^{ab}(t) &= \left(\sum_{i \in X_l^a} p_i(t) p_{bu}(t) \sum_{j \in X_m^a} \lambda_{ij} \right) \Big/ \sum_{i \in X_l^a} p_i(t) p_{bu}(t) \\ &= \left(\sum_{i \in X_l^a} p_i(t) \sum_{j \in X_m^a} \lambda_{ij} \right) \Big/ \sum_{i \in X_l^a} p_i(t) \\ &= \lambda_{lm}^{a}(t) \end{aligned}$$

The time specific equivalent transfer rate, therefore, holds unconditionally if the subsystems being combined are independent. The evaluation of these rates is, of course, a separate problem.

(2) If, however, the subsystems are not independent, the Equality (5.7) can hold if

(i) $\sum_{j \in X_m^a} \lambda_{ij}$ is the same for any i in which case

$$\lambda_{lm}^{ab}(t) = \left(\sum_{j \in X_m^a} \lambda_{ij}\right) \sum_{i \in X_l^a} p_i^u(t) \Big/ \sum_{i \in X_l^a} p_i^u(t)$$

$$= \sum_{j \in X_m^a} \lambda_{ij}$$

$$= \lambda_{lm}^a$$

This is the same condition as previously specified for mergeability.

(ii) Alternatively if for a given initial state probability vector

$$p_1^u(t) = p_2^u(t) = \ldots = p_i^u(t) \qquad \text{which implies}$$

$$p_1(t) = p_2(t) = \ldots = p_i(t)$$

i.e., if the states being lumped have the same time specific probability

$$\lambda_{lm}^{ab}(t) = \lambda_{lm}^a$$

$$= \sum_{i \in X_l^a} \sum_{j \in X_m^a} \lambda_{ij}/n_l$$

where n_l = The number of states in subset X_l^a

Steady state is a special case of transient analysis with $t \to \infty^+$. The analysis for steady state conditions of mergeability is the same as that for the transient case with the following differences:

1. The steady state equivalent transfer rate is time invariant.
2. The steady state probabilities are independent of the initial conditions and therefore the steady state equivalent transfer rate is independent of such a restriction.

The results of the mergeability analysis are summarized below.

1. If $\Sigma\lambda_{ij}$ $_{j \epsilon X_m^a}$ is the same for any $i \epsilon X_l^a$, then the equivalent transfer rate is time invariant and independent of the initial condition. When this condition is satisfied between all the subsets taken in pairs, the Markov process of the subsystem S_a is said to be mergeable into these disjoint subsets. The equivalent transfer rates so determined hold when this subsystem is combined with another subsystem S_b, provided the interstate transition information contained in the lumped states does not change by virtue of this combination.
2. If for the given initial condition, the probabilities of the states in a subset are equal, then these states are mergeable and the equivalent transition rate holds when this subsystem is combined with another subsystem, provided the initial condition is not violated. In the steady state case, the initial condition does not affect the conclusion.
3. For independent subsystems, the states of the subsystem may be lumped into any desired disjoint subsets. The equivalent transition rate is unaltered by combination but because of the computation effort required, this does not seem to be of much significance in the time specific analysis. In steady state analysis, this can facilitate the calculation of the frequency index. A familiar example is the lumping together of the identical capacity outage states in a generation system model before combining it with the load model.

Example 5.1: The concepts outlined in the previous section can be illustrated for the steady state condition using the simple system shown in Fig. 5.1.

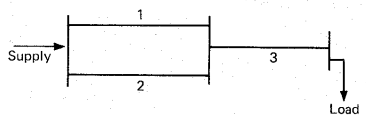

Fig. 5.1 A simple three unit series-parallel system

The system consists of two lines 1 and 2 in parallel. This combination is then in series with component 3. Both lines 1 and 2 have identical failure and repair rates λ and μ and each is capable of supplying the full load. The failure and repair rates of line 3 are λ_3 and μ_3 respectively. The system is now assumed to consist of subsystem S_a of lines 1 and 2 and the subsystem S_b of line 3. The state transition diagrams of the two subsystems taken individually are shown in Figs. 5.2 (a) and (b) respectively. On combining S_a and S_b, there will be a total of

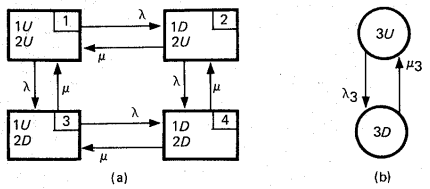

Fig. 5.2 (a) The state transition diagram of Subsystem S_a
(b) The state transition diagram of Subsystem S_b

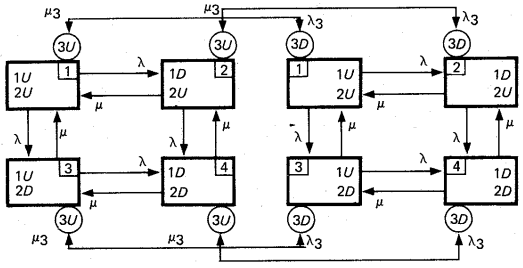

Fig. 5.3 The state transition diagram after combining S_a and S_b

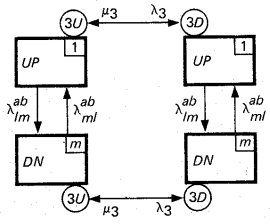

Fig. 5.4 The state transition diagram after combining reduced S_a with S_b

eight states. These can be written as

$$(3U)1, (3U)2, (3U)3, (3U)4$$

$$(3D)1, (3D)2, (3D)3, (3D)4$$

The condition inside the parenthesis is that of S_b and the outside numbers refer to the state numbers of S_a. Only one line out of $\{1,2\}$ is required for successful operation and therefore the behaviour of S_a can be represented by a two state component in which state l corresponds to states $\{1,2,3\}$ and state m to state $\{4\}$ of the original state space. The equivalent transition rates are

$$\lambda_{lm} = \frac{\lambda(p_2 + p_3)}{p_1 + p_2 + p_3}$$

and

$$\lambda_{ml} = 2\mu$$

The question to be examined is, do these equivalent transfer rates hold after combining S_a and S_b? This problem is considered for both complete and restricted repair facilities.

1 *Complete Repair Facilities*

When each component can be repaired independently, the two subsystems are independent. It can also be observed that when the two subsystems are independent, the interstate transition modes remain unchanged after interaction. Now, if merging is to be valid, the equivalent transition rate λ_{lm}^{ab}, i.e. from $(3U)l$ to $(3U)m$ should be the same as λ_{lm}, i.e. from l to m. Examining the state transition diagram in Fig. 5.3

$$\lambda_{lm}^{ab} = \frac{(p_2^u + p_3^u)\lambda}{p_1^u + p_2^u + p_3^u}$$

Here the superscript refers to the condition of system b. Since the subsystems are independent

$$\lambda_{lm}^{ab} = \frac{(p_2 p_{3u} + p_3 p_{3u})\lambda}{p_1 p_{3u} + p_2 p_{3u} + p_3 p_{3u}} = \frac{(p_2 + p_3)\lambda}{p_1 + p_2 + p_3}$$

$$= \lambda_{lm}$$

It can be concluded from the above discussion that when S_a and S_b are independent, subsystem S_a can be represented by a two state component having the equivalent transition rates determined by Equation (5.1). This is the basis of the network reduction technique described in the previous chapter. The reason for assuming the subsystems to be independent can now be more fully understood.

2 *Restricted Repair Facilities*

Now suppose that only two lines can be repaired at a time, then in the event of failure of all the three lines, state (3*D*)4, in Fig. 5.3, the repair of one line out of $\{1,2\}$ must wait for the first repair. Referring to Fig. 5.4, λ^{ab}_{ml} from (3*U*)*m* to (3*U*)*l* is still 2μ but λ^{ab}_{ml} from (3*D*)*m* to (3*D*)*l* is now μ. The systems are no longer independent and therefore $p^u_i \neq p'_i.p^u_3$, and consequently $\lambda^{ab}_{lm} \neq \lambda_{lm}$. Merging into the above groupings is not possible.

If, however, states $\{2,3\}$ are merged to give the equivalent state *l* and if $\{1\}$ and $\{4\}$ are denoted by *n* and *m* respectively, the conditions of mergeability are satisfied and it can be seen that

$$\lambda^{ab}_{ln} = \lambda_{ln} = \mu$$

and

$$\lambda^{ab}_{lm} = \lambda_{lm} = \lambda$$

Merging into these states is therefore still possible.

Components Subject to Fluctuating Environment

The following considers the application of conditions of mergeability to a system consisting of components exposed to a two state fluctuating environment. The two states are designated as *N* and *S* states and the durations of these states are assumed to be exponentially distributed. The component failure and repair rates are constant but depend on the environment. Failure and repair of the components are independent in a given environment but the exposure to the common environment introduces the element of interdependence and the probabilities cannot be found by a simple product rule. The conditions of mergeability developed in this chapter will now be applied to this system.

The system is assumed to be divided into a number of subsystems out of which only subsystems S_a and S_b are considered for the sake of convenience. These subsystems are assumed to contain n_a and n_b number of components respectively. Considering S_a, it can exist in 2^{n_a} number of states in each of the environmental states. The component configuration of each state in either of the two environment states is the same but the interstate transition rates in the two weather states are different. The transition rate from a system state in the *N* environment condition to the corresponding system state in the *S* environment condition is taken as *n* and in the reverse direction as *m*. The states in the *N* environment may be grouped into subsets $X^a_1, X^a_2, \ldots, X^a_p$ and those in the *S* environment as $X^{as}_1, X^{as}_2, \ldots, X^{as}_p$. The combination of states in X^a_i and X^{as}_i is the same. Since the reduced state space of S_a is to be combined with the reduced state space of S_b which is also exposed to the same fluctuating environment, the states in the *N* and *S* environment cannot be lumped together. The equivalent

transfer rates can be found by the application of Equation (5.1). It is obvious that the transition rate from X^a_i to X^{as}_i is *n* and that from X^{as}_i to X^a_i is *m*. The equivalent transfer rate from X^a_l to X^a_m is given by

$$\lambda^a_{lm} = \left(\sum_{i \in X^a_l} p_i \sum_{j \in X^a_m} \lambda_{ij}\right) \Big/ \sum_{i \in X^a_l} p_i$$

where

p_i = The probability of being in the *i*th state in *N* environment

and

λ^a_{lm} = The equivalent transfer rate from the lumped states of subsets X^a_l to those of X^a_m.

The same treatment holds for the states in the *S* environment. After merging together the states in each subset, there will be *p* equivalent states in each environment condition in the subsystem S_a, one equivalent state corresponding to one subset. These equivalent states may be indicated by $a_1, a_2, \ldots, a_p$ in the *N* environment condition, and $a^s_1, a^s_2, \ldots, a^s_p$ in the *S* environment condition. Now suppose that this subsystem is combined with the other subsystem S_b with equivalent states $\{b_1, b_2, \ldots, b_q\}$ and $\{b^s_1, b^s_2, \ldots, b^s_q\}$ in the *N* and *S* environment condition. In the combined system there will be $p \times q$ states in each environment condition. This combination for the *N* environment is shown below.

$$\begin{array}{llll} (b_1)a_1, & (b_1)a_2, & \ldots, & (b_1)a_p \\ (b_2)a_1, & (b_2)a_2, & \ldots, & (b_2)a_p \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ (b_q)a_1, & (b_q)a_2, & \ldots, & (b_q)a_p \end{array}$$

Since the two subsystems are exposed to the same fluctuating environment, it can be easily seen that the equivalent transfer rate from $(b_i)a_j$ to $(b^s_i)a^s_j$ is *n* and that from $(b^s_i)a^s_j$ to $(b_i)a_j$ is *m*. For steady state mergeability, the transfer rate from $(b_u)a_l$ to $(b_u)a_m$ should be the same as that from a_l to a_m, i.e.,

$$\lambda^{ab}_{lm} = \lambda^a_{lm}$$

By the applications of the mergeability conditions the following conclusions can be drawn:

1. If S_a and S_b are independent, the equivalent transfer rates hold unconditionally, i.e. the states of the subsystem may be lumped into any desired disjoint subsets. When S_a and S_b are exposed to the same fluctuating environment, the independence is possible if either S_a or S_b or both subsystems have components whose failure and repair rates are the same in both environment

states, i.e. these rates are environment independent.

2. In subsystems which are not independent, the states can be lumped together if, and only if, either

(i) All the states being merged have the same availability, or

(ii) The sum of transfer rates from state $i \in X^a$ to all states $j \in X_m^a$, i.e., $\sum_{j \in X_m^a} \lambda_{ij}$ is the same for all i.

The conditions of steady state mergeability restrict the scope of model reduction when the subsystems are not independent but even in this restricted sense model reduction can be of considerable help in large sized networks.

One obvious application is when the elements of a subsystem have identical failure and repair rates. An n identical element subsystem can be lumped into $2(n+1)$ states. For example, a four element subsystem will have the following groups of identical states.

Group number	*Number of elements failed*	*Number of identical states in group*
1	0	$\binom{4}{0} = 1$
2	1	$\binom{4}{1} = 4$
3	2	$\binom{4}{2} = 6$
4	3	$\binom{4}{3} = 4$
5	4	$\binom{4}{4} = 1$
	Total	16

The 16 states for each environment state can be lumped into five equivalent states. Therefore, the subsystem having 32 states can be adequately represented by 10 states. This application to identical elements subsystems becomes important when it is realized that parallel facilities normally have identical failure and repair parameters.

The merging technique can be illustrated by application to the simple system shown in Fig. 5.5. The data for this system is given below. The N environment corresponds to normal weather and the S environment to stormy or adverse weather.

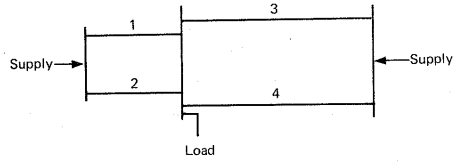

Fig. 5.5 A simple transmission network Mean duration N = 200 hours
Mean duration S = 1.5 hours

Percentage of failures during S environment = 20%

Components	*Average failure rate per year*	*Mean down time hours*
1,2	0·5	5.0
3,4	1·0	10·0

This system can be split into two subsystems A and B having $\{1,2\}$ and $\{3,4\}$ components. The state transition diagram of system i (= a or b) is shown in Fig. 5.6.

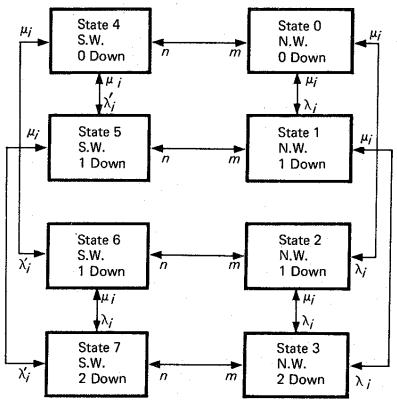

Fig. 5.6: Two identical unit state transition diagram for a two state fluctuating environment.

The following notation is used

$m = \frac{1}{S}$ = Transition rate from Stormy state (SW) to Normal state (NW).

$n = \frac{1}{N}$ = Transition rate from Normal state to Stormy state.

μ_i = The mean repair rate of a component in the ith system. The same repair rate is assumed in normal and stormy states.

λ_i = Normal state failure rate of a component in the ith system.

λ_i' = Stormy state failure rate of a component in the ith system.

The average failure rate λ_{av} is given by

$$\lambda_{av} = \lambda\left[\frac{N}{N+S}\right] + \lambda'\left[\frac{S}{N+S}\right] \tag{5.7}$$

If the number of failures in the stormy weather is x percent

$$\frac{\lambda' S}{\lambda N + \lambda' S} = \frac{x}{100} \tag{5.8}$$

The value of λ and λ' can be determined using Equations (5.7) and (5.8).

Subsystem Reduction

In subsystem i, states 1, 2 and 5, 6 are identical and therefore have equal availabilities. Condition 2(i) is thus satisfied and these states can be lumped together. The reduced model of subsystem A is shown in Fig. 5.7. This reduced model can be combined with the reduced model of system B. The resulting state transition diagram is shown in Fig. 5.8. In this diagram, $N\,W$, $S\,W$, stand for normal weather and stormy weather states and XI, ($X = A$ or B and $I = 0, 1, 2$) stands for system X with I components down. It can be seen that in the

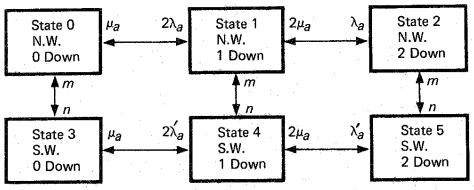

Fig. 5.7 The reduced model of subsystem A

reduced model there are only 18 states as against 32 states in the original model.

The steady state equations can be written in the form

$$TP = B \tag{5.9}$$

where

$$T = \left[\begin{array}{c|c|c} T_{11} & T_{12} & T_{13} \\ \hline T_{21} & T_{22} & T_{23} \\ \hline T_{31} & T_{32} & T_{33} \end{array}\right]$$

where

$$T_{12} = \mu_b\,[\mathrm{I}]\ (6 \times 6 \text{ matrix})$$

$$T_{13} = 0 \quad (6 \times 6 \text{ matrix})$$

$$T_{21} = \left[\begin{array}{c|c} 2\lambda_b\,[\mathrm{I}] & 0 \\ \hline 0 & 2\lambda'_b\,[\mathrm{I}] \end{array}\right] \quad (6 \times 6 \text{ matrix})$$

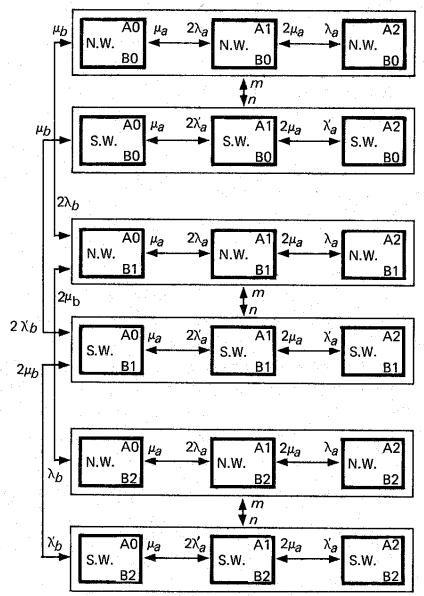

Fig. 5.8 The state transition diagram of the system shown in Fig. 6.1

$$T_{23} = 2 \times T_{12} \text{ (6 x 6 matrix)}$$

$$T_{31} = 0 \qquad \text{(6 x 6 matrix)}$$

$$T_{32} = \begin{bmatrix} \lambda_b[\mathrm{I}] & 0 \\ 0 & \lambda'_b[\mathrm{I}] \end{bmatrix}$$

$$T_{11} = A - T_{21}$$

$$T_{22} = A - T_{12} - T_{32}$$

and $T_{33} = A - T_{23}$

where

$$A = \begin{bmatrix} -(2\lambda_a + n) & \mu_a & 0 & m & 0 & 0 \\ 2\lambda_a & -(\mu_a + \lambda_a + n) & 2\mu_a & 0 & m & 0 \\ 0 & \lambda_a & -(2\mu_a + n) & 0 & 0 & m \\ n & 0 & 0 & -2(\lambda'_a + m) & \mu_a & 0 \\ 0 & n & 0 & 2\lambda'_a & -(\mu_a + \lambda'_a + m) & 2\mu_a \\ 0 & 0 & n & 0 & \lambda'_a & -(2\mu_a + m) \end{bmatrix}$$

B = A column matrix with zero entries

and

$$P = \{p_{(0,0)N}, p_{(0,1)N}, \ldots, p_{(2,0)S}, p_{(2,1)S}, p_{(2,2)S}\}$$

In the triple subscript of p the first two terms inside the parenthesis indicate the state of the components in subsystems B and A respectively, e.g. (0,1) denotes no component down in subsystem B and one component down in subsystem A. The subscript outside the parenthesis indicates the environment state. Any seventeen equations of (5.9) with

$$\sum_{i=0}^{2} \sum_{j=0}^{2} \sum_{k=N,S} p_{(i,j)k} = 1{\cdot}0$$

may be solved to obtain the vector P of steady state availabilities. The results obtained by solving this set of equations are shown in Table 5.1. In column one, the first number inside the parentheses indicates the number of failed components of subsystem A and the second in B. In column (2), the availabilities of the normal and stormy states are merged together. Column (3) gives the number of identical states in each environment state in the original model. The values of column (4) are obtained by dividing the values in column (2) by those in column (3). The results in column (5) are obtained by the analysis of the complete model of 32 states. It can be seen that the results in columns (4) and (5) have only very slight differences due to rounding off. Complete information about the 32 states of the original system can thus be obtained by analyzing the reduced model of 18 states.

Table 5.1 *Comparison of the results obtained from the reduced model and the original model of the system shown in Fig. 1*

Description of the lumped state	*Availability of the lumped state*	*Number of states lumped in each environment state*	*Availability of one identical state*	*Availability of one identical state obtained from the non-reduced model*
(1)	(2)	(3)	(4)	(5)
(0,0)	0·997155	1	0·997155	0·997156
(0,1)	$0{\cdot}567732\times10^{-3}$	2	$0{\cdot}283866\times10^{-3}$	$0{\cdot}283866\times10^{-3}$
(0,2)	$0{\cdot}172580\times10^{-6}$	1	$0{\cdot}172580\times10^{-6}$	$0{\cdot}172580\times10^{-6}$
(1,0)	$0{\cdot}227221\times10^{-2}$	2	$0{\cdot}113610\times10^{-2}$	$0{\cdot}113611\times10^{-2}$
(1,1)	$0{\cdot}233858\times10^{-5}$	4	$0{\cdot}584645\times10^{-6}$	$0{\cdot}584645\times10^{-6}$
(1,2)	$0{\cdot}203460\times10^{-8}$	2	$0{\cdot}101730\times10^{-8}$	$0{\cdot}101729\times10^{-8}$
(2,0)	$0{\cdot}212768\times10^{-5}$	1	$0{\cdot}212768\times10^{-5}$	$0{\cdot}212769\times10^{-5}$
(2,1)	$0{\cdot}616139\times10^{-8}$	2	$0{\cdot}308069\times10^{-8}$	$0{\cdot}308070\times10^{-8}$
(2,2)	$0{\cdot}961824\times10^{-11}$	1	$0{\cdot}961824\times10^{-11}$	$0{\cdot}961873\times10^{-11}$

State Space Truncation

It has been seen that the state space can be reduced by merging certain groups of states. Another technique is by truncating the state space, i.e. by neglecting the states whose contribution to the measures of system reliability is insignificant. In systems consisting of independent components, the probability of each state can be calculated individually by the product of the individual component probabilities. The states required for determining the reliability measures are selected, their probabilities calculated and the reliability measures

obtained. The system states which make a negligible contribution to the final results can be neglected.

When dependent transition modes are involved, the system state probabilities cannot be obtained directly and the set of differential or linear algebraic equations must be solved depending on whether time specific or steady state solutions are required. Consider first the steady state condition.

The philosophy behind truncation may be understood by examining the following equation for calculating the probability of the ith state

$$p_i = \sum_{k \in X^-} p_k \lambda_{ki} \Big/ \sum_{k \in X^-} \lambda_{ik}$$

The contribution to p_i by a state $k \neq i$ is

$$p_k \lambda_{ki} \Big/ \sum_{k \in X^-} \lambda_{ik}$$

i.e. the frequency of encountering i from k divided by the total transition rate out of i. Therefore if the states having low probability are deleted, the probability of state i will not be significantly effected. The states have of course to be deleted prior to solving the set of linear equations. The procedure amounts to assuming that the deleted states have a probability equal to zero. Denoting the set of deleted states by X_T, the probability of this subset if there were no truncation is $p_T = \Sigma p_i$. Since the probability of the rest of the state space is $i \epsilon X_T$
now one, i.e.

$$\sum_{i \in (X - X_T)} p_i = 1$$

the probability p_T will be distributed over the states $i \epsilon (X\text{-}X_T)$ where X is the system state space. If p_T is small, then the probability distribution of the rest of the states will not be significantly affected. The success of the truncation method depends upon selecting low probability states for truncation. The following consideration should be kept in mind while employing truncation.

1. The probability p_i, $i \epsilon X_T$ is less than p_j, $j \epsilon (X\text{-}X_T)$. In words, the biggest probability in the truncated subset should be less than the smallest probability of the remaining state space. In systems consisting of two state components, this is not hard to achieve. The state space may be divided into subsets, each subset having states of a certain level of coincident failures. For a system of n identical components there will be $(n+1)$ subsets. These subsets will have the following states:

Subset number	*States description*
1	All components up
2	One component down
.	.
.	.
.	.
$n+1$	n components down

An arbitrary level of truncation should be first selected; for example the states having three or more than three coincident failures can be truncated. The computation can then be repeated by including the next subset, i.e. the states having three coincident failures. If the new values are not significantly different from the previous ones, the computation can be stopped, otherwise one more subset should be included and the computation repeated.

If the units have derated levels or if there are more than one down state, for example, repair and switching states, the same procedure should be adopted with the addition that the states like switching associated with the repair state should also be retained.

In the state space truncation technique, the probabilities of the states adjacent to the truncation boundary are affected the most and the effect decreases when moving away from the boundary.

2. After the states have been truncated, the state transition diagram should be examined to see if the process of truncation has generated any absorbing states. Since the computer program generates only the transition rate matrix, the absorbing states can be located by examining this matrix. An absorbing state will have transitions into it but not out of it. The ijth element of the transition rate matrix gives the transition rate from state i to state j. Therefore if the ith row is empty, this means that the ith state is absorbing. Either the absorbing state should be deleted or the states where truncation has generated this absorbing state should be retained.

Example: Consider a transmission system consisting of five links subjected to a common two state fluctuating environment, normal and stormy weather are 32 states in each environment and a total of 64 states. The distribution of states in each environment state is shown below.

Subset number	1	2	3	4	5	6
Number of elements failed	0	1	2	3	4	5
Number of states	1	5	10	10	5	1

If, however, the probabilities of failure of three or more components can be assumed to be zero, the states in subsets 4–6 can be ignored and the matrix of transition rates is reduced from 64 x 64 to 32 x 32. A number of studies were

performed on this system to test the sensitivity of the probability distribution over the various system states to truncation. The results indicate that this distribution is relatively insensitive to truncation. The results of a sample study are shown in Table 5.2 along with the relevant component data. Table 5.2A gives the availabilities of various states in the original model under different component and weather state parameters. Identical environment states are shown merged together. Tables 5.1B to 5.1E show the effects of truncation. The limit of components on outage is indicated by *MC*, e.g. *MC* = 4 means that the probability of failure of components more than four is zero. As can be seen from Tables 5.1B – 5.1E, the percentage error is almost negligible. This error, however, increases as *MC* decreases. It is obvious that the larger the number of components, the less sensitive is the probability distribution to conditional truncation. For a large transmission system, the system states can, therefore, be conditionally truncated without causing any significant error.

Sequential Truncation

Sequential truncation can be described as the process of building the reliability model by adding components or subsystems one by one and deleting the low probability states at each step. This method consumes more computation time than direct state space truncation but it is more manageable. In direct truncation, the decision to delete states has to be made prior to the solution of the state probabilities. In sequential truncation, the state probabilities are calculated at each step and the states with probabilities less than a reference value are deleted. The assumption, which is generally valid is that the probability of a given state will be decreased after another component has been added to the system. Assume that at a particular step the system has q states designated $S_1, S_2, \ldots, S_q$, and a component having p states, designated $C_1, C_2, \ldots, C_p$ is added. If this component were independent from the system which has been built up to this point, then there would be $q \times p$ states in the resulting system, there being p states for every state of the system up to this point. This may be represented as

$$\begin{array}{cccc} S_1C_1 & S_1C_2 & \ldots & S_1C_p \\ S_2C_1 & S_2C_2 & \ldots & S_2C_p \\ S_3C_1 & S_3C_2 & \ldots & S_3C_p \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ S_qC_1 & S_qC_2 & \ldots & S_qC_p \end{array}$$

Table 5.2 *Studies of the sensitivity of the availabilities of various systems to be conditional truncation of system states*

Number of identical components = 5
Average failure rate = 0·5/year
Normal weather mean duration = 200 hours
Stormy weather mean duration = 1·5 hours

A: Availabilities of all possible states

Group number	*Number of identical components states in the group down*		*Availability of one identical state in the group*			% Failures during S.W. = 20			% Failures during S.W. = 80		
			γ = 5 hours	γ = 10 hours	γ = 22·5 hours	γ = 5 hours	γ = 10 hours	γ = 22·5 hours	γ = 5 hours	γ = 10 hours	γ = 22·5 hours
1	0	1	0·998576	0·997154	0·993611	0·998598	0·997203	0·993729			
2	1	5	$0{\cdot}284518\times10^{-3}$	$0{\cdot}568107\times10^{-3}$	$0{\cdot}127351\times10^{-2}$	$0{\cdot}277285\times10^{-3}$	$0{\cdot}551913\times10^{-3}$	$0{\cdot}123463\times10^{-2}$			
3	2	10	$0{\cdot}173558\times10^{-6}$	$0{\cdot}531820\times10^{-6}$	$0{\cdot}213447\times10^{-5}$	$0{\cdot}158411\times10^{-5}$	$0{\cdot}366977\times10^{-5}$	$0{\cdot}960027\times10^{-5}$			
4	3	10	$0{\cdot}354657\times10^{-9}$	$0{\cdot}123808\times10^{-8}$	$0{\cdot}595186\times10^{-8}$	$0{\cdot}181082\times10^{-7}$	$0{\cdot}517200\times10^{-7}$	$0{\cdot}157760\times10^{-6}$			
5	4	5	$0{\cdot}122932\times10^{-11}$	$0{\cdot}535826\times10^{-11}$	$0{\cdot}287183\times10^{-10}$	$0{\cdot}262773\times10^{-9}$	$0{\cdot}985960\times10^{-9}$	$0{\cdot}368171\times10^{-8}$			
6	5	1	$0{\cdot}509552\times10^{-14}$	$0{\cdot}302236\times10^{-13}$	$0{\cdot}198045\times10^{-12}$	$0{\cdot}439865\times10^{-11}$	$0{\cdot}227601\times10^{-10}$	$0{\cdot}108524\times10^{-9}$			

γ = mean down time

B: Percentage difference from the exact values, $MC = 4$

Group number	*Number of components down*	*Number of identical states in the group*	*% Difference from the exact values in Table A*					
			% Failures during S.W. = 20			% Failures during S.W. = 80		
			γ = 5 hours	γ = 10 hours	γ = 22·5 hours	γ = 5 hours	γ = 10 hours	γ = 22·5 hours
1	0	1	0·0	0·0	0·0	0·0	0·0	0·0
2	1	5	0·0	0·0	0·0	0·0	0·0	0·0
3	2	10	0·0	0·0	0·0	0·000001	0·000001	0·0
4	3	10	0·000013	0·000005	0·000009	0·000036	0·000050	0·000009
5	4	5	0·004525	0·000022	0·001343	0·002240	0·005604	0·001617

C: Percentage difference from the exact values, $MC = 3$

Group number	*Number of components down*	*Number of identical states in the group*	*% Difference from the exact values in Table A*					
			% Failures during S.W. = 20			% Failures during S.W. = 80		
			γ = 5 hours	γ = 10 hours	γ = 22·5 hours	γ = 5 hours	γ = 10 hours	γ = 22·5 hours
1	0	1	0·0	0·0	0·0	0·0	0·0	0·000002
2	1	5	0·0	0·0	0·0	0·000004	0·000001	0·000003
3	2	10	0·000009	0·000003	0·000003	0·000165	0·000035	0·000042
4	3	10	0·002385	0·001277	0·000535	0·007575	0·004306	0·005267

D: Percentage difference from the exact values, $MC = 2$

Group number	*Number of components down*	*Number of identical states in the group*	*% Difference from the exact values in Table A*					
			% Failures during S.W. = 20			% Failures during S.W. = 80		
			γ = 5 hours	γ = 10 hours	γ = 22·5 hours	γ = 5 hours	γ = 10 hours	γ = 22·5 hours
1	0	1	0·0	0·000001	0·000006	0·000018	0·000052	0·000159
2	1	5	0·000001	0·000002	0·000001	0·000065	0·000062	0·000175
3	2	10	0·000466	0·000557	0·000928	0·007762	0·009008	0·008023

E: Percentage difference from the exact values, $MC = 1$

Group number	*Number of components down*	*Number of identical states in the group*	*% Difference from the exact values in Table A*					
			% Failures during S.W. = 20			% Failures during S.W. = 80		
			γ = 5 hours	γ = 10 hours	γ = 22·5 hours	γ = 5 hours	γ = 10 hours	γ = 22·5 hours
1	0	1	0·000174	0·000534	0·002139	0·001583	0·003695	0·009718
2	1	5	0·000226	0·000194	0·002276	0·014940	0·013217	0·016551

The probability of state S_iC_j will be the product of the probabilities of states S_i and C_j. Since the probability of state C_j cannot be greater than one, the probability of S_iC_j will be always less than the probability of S_i. Therefore if the probability of state S_iC_j is less than the reference value, the probability of S_iC_j will also be less than the reference value.

When, however, dependent transition modes are involved, some of the above stage combinations may not exist. The probabilities of the resulting states will, however, be generally less than those of the states prior to combination. The method of sequential truncation will be illustrated by application to an example system.

The system consists of three identical railway stations as shown in Fig. 5.9. Each station has a station lane on which platform facilities are provided for passenger disembarkation and a through expressway without any platform facilities. When the station lane and through expressway are both down, traffic cannot pass through and the system is considered failed. It is assumed that under this condition, no further component failure takes place. The guideway is assumed to be perfectly reliable. The results of this subsystem are to be combined with the other subsystems and therefore the probabilities and frequencies of various states of this system are to be determined. The failure and repair rate of the station lane are denoted by λ_s and μ_s and those of the expressway by λ_e and μ_e. The following numerical values have been used

$$\text{Mean Up Time of the station lane} = \frac{1}{\lambda_s} = 800 \text{ Hours}$$

$$\text{Mean Down Time of the station lane} = \frac{1}{\mu_s} = 2 \text{ Hours}$$

$$\text{Mean up time of the expressway} = \frac{1}{\lambda_e} = 1000 \text{ Hours}$$

$$\text{Mean down time of the expressway} = \frac{1}{\mu_e} = 2 \text{ hours}$$

As the guideway is considered perfectly reliable, it can be left out of the analysis. The state transition diagram of a station is shown in Fig. 5.10, U stands for the up state, i.e. when both lanes are up. D means that the station

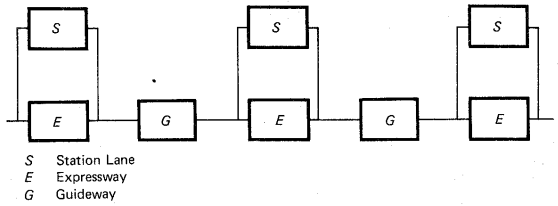

Fig. 5.9. The subsystem functional diagram

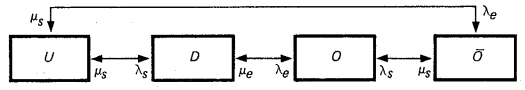

Fig. 5.10 The state transition diagram of a station

lane is down but the traffic can pass through the expressway. O denotes the complete station outage, i.e. both lanes are out and O stands for partial outage, i.e. the station lane is working but the through expressway is down. These states can be represented in the computer as shown in Table 5.3A. The addition of one more station is shown in Table 5.3B. For each state of the component, there is the set of system states of Table 5.3A except that state (3,3) is an impossible state since it means that the two stations are completely out. This is not possible as the exposure to failure is reduced to zero as soon as one station is completely out. The system states in Table 5.3B are numbered in the serial order. The numbers in brackets indicate the combination, the first number indicating the state number of the system before addition and the second indicating the state number of the component being added. Identical states can now be grouped together and the resulting description is shown in Table 5.3C. The states with probabilities less than 10^{-5} are now deleted and the resulting description is given in Table 5.3D. The state numbers are the serial numbers and have no relationship to the state numbers in Table 5.3C.

When the third station is added, the resulting states are shown in Table 5.3E and the states after merging identical states are given in Table 5.3F. The state probabilities are also indicated. If more stations are to be added, then the states with probabilities less than 10^{-5} can again be deleted and the procedure repeated. The exact results, i.e. without any truncation, are shown in Table 5.4

and it can be seen that the results are almost identical. In general, the results are slightly affected depending on the reference probability value employed for truncation.

Table 5.3

A. Model of a single station

System state	*Identical states from B*	*Number of stations in state* U	D	0	$\bar{0}$	*Probability*
1		1	0	0	0	
2		0	1	0	0	
3		0	0	1	0	
4		0	0	0	1	

B. Addition of a station

1	(1,1)	2	0	0	0	
2	(2,1)	1	1	0	0	
3	(3,1)	1	0	1	0	
4	(4,1)	1	0	0	1	
5	(1,2)	1	1	0	0	
6	(2,2)	0	2	0	0	
7	(3,2)	0	1	1	0	
8	(4,2)	0	1	0	1	
9	(1,3)	1	0	1	0	
10	(2,3)	0	1	1	0	
11	(4,3)	0	0	1	1	
12	(1,4)	1	0	0	1	
13	(2,4)	0	1	0	1	
14	(3,4)	0	0	1	1	
15	(4,4)	0	0	0	2	

C. Merging of identical states

1	1	2	0	0	0	$0{\cdot}991051\times10^{0}$
2	2,5	1	1	0	0	$0{\cdot}495525\times10^{-2}$
3	3,9	1	0	1	0	$0{\cdot}992536\times10^{-5}$
4	4,12	1	0	0	1	$0{\cdot}396420\times10^{-2}$
5	6	0	2	0	0	$0{\cdot}618994\times10^{-5}$
6	7,10	0	1	1	0	$0{\cdot}165058\times10^{-7}$
7	8,13	0	1	0	1	$0{\cdot}990308\times10^{-5}$
8	11,14	0	0	1	1	$0{\cdot}132036\times10^{-7}$
9	15	0	0	0	2	$0{\cdot}396090\times10^{-5}$

D. Truncation of states with probabilities less than 10^{-5}

System state	*Number of stations in state* U	D	0	$\bar{0}$	*Probability*
1	2	0	0	0	$0{\cdot}991051\times10^{0}$
2	1	1	0	0	$0{\cdot}495525\times10^{-2}$
3	1	0	1	0	$0{\cdot}992536\times10^{-5}$
4	1	0	0	1	$0{\cdot}396420\times10^{-2}$
5	0	2	0	0	$0{\cdot}618994\times10^{-5}$
6	0	1	0	1	$0{\cdot}990308\times10^{-5}$
7	0	0	0	2	$0{\cdot}396090\times10^{-5}$

E. Addition of the third station

System state		*Number of stations in state* U	D	0	$\bar{0}$	
1	(1,1)	3	0	0	0	
2	(2,1)	2	1	0	0	
3	(3,1)	2	0	1	0	
4	(4,1)	2	0	0	1	
5	(5,1)	1	2	0	0	
6	(6.1)	1	1	0	1	
7	(7,1)	1	0	0	2	
8	(1,2)	2	1	0	0	
9	(2,2)	1	2	0	0	
10	(3,2)	1	1	1	0	
11	(4,2)	1	1	0	1	
12	(5,2)	0	3	0	0	
13	(6,2)	0	2	0	1	
14	(7,2)	0	1	0	2	
15	(1,3)	2	0	1	0	
16	(2,3)	1	1	1	0	
17	(4,3)	1	0	1	1	(3,3) not possible
18	(5,3)	0	2	1	0	
19	(6,3)	0	1	1	1	
20	(7,3)	0	0	1	2	
21	(1,4)	2	0	0	1	
22	(2,4)	1	1	0	1	
23	(3,4)	1	0	1	1	
24	(4,4)	1	0	0	2	
25	(5,4)	0	2	0	1	
26	(6,4)	0	1	0	2	
27	(7,4)	0	0	0	3	

F. Merging of identical states

System state	*Identical states from E*	*Number of stations in state* U	D	0	$\bar{0}$	*Probability*
1	1	3	0	0	0	0·986606
2	2,8	2	1	0	0	$0{\cdot}739954\times10^{-2}$
3	3,15	2	0	1	0	$0{\cdot}148435\times10^{-4}$
4	4,21	2	0	0	1	$0{\cdot}591964\times10^{-2}$
5	5,9	1	2	0	0	$0{\cdot}184865\times10^{-4}$
6	6,11,22	1	1	0	1	$0{\cdot}295760\times10^{-4}$
7	7,24	1	0	0	2	$0{\cdot}118294\times10^{-4}$
8	10,16	1	1	1	0	$0{\cdot}493508\times10^{-7}$
9	12	0	3	0	0	$0{\cdot}153901\times10^{-7}$
10	13,25	0	2	0	1	$0{\cdot}369310\times10^{-7}$
11	14,26	0	1	0	2	$0{\cdot}295407\times10^{-7}$
12	17,23	1	0	1	1	$0{\cdot}394773\times10^{-7}$
13	18	0	2	1	0	$0{\cdot}461670\times10^{-10}$
14	19	0	1	1	1	$0{\cdot}738569\times10^{-10}$
15	20	0	0	1	2	$0{\cdot}295387\times10^{-10}$
16	27	0	0	0	3	$0{\cdot}787643\times10^{-8}$

Table 5.4 *Model of three stations without truncation*

System state	*State number as in 5.3F*	*Number of stations in state* U	D	0	$\bar{0}$	*Probability*
1	1	3	0	0	0	0·986606
2	2	2	1	0	0	$0{\cdot}739954\times10^{-2}$
3	3	2	0	1	0	$0{\cdot}148435\times10^{-4}$
4	4	2	0	0	1	$0{\cdot}591964\times10^{-2}$
5	5	1	2	0	0	$0{\cdot}184865\times10^{-4}$
6	8	1	1	1	0	$0{\cdot}493508\times10^{-7}$
7	6	1	1	0	1	$0{\cdot}295760\times10^{-4}$
8	12	1	0	1	1	$0{\cdot}394773\times10^{-7}$
9	7	1	0	0	2	$0{\cdot}118294\times10^{-4}$
10	9	0	3	0	0	$0{\cdot}153901\times10^{-7}$
11	13	0	2	1	0	$0{\cdot}461824\times10^{-10}$
12	10	0	2	0	1	$0{\cdot}369310\times10^{-7}$
13	14	0	1	1	1	$0{\cdot}738538\times10^{-10}$
14	11	0	1	0	2	$0{\cdot}295407\times10^{-7}$
15	Deleted	0	1	2	0	$0{\cdot}123121\times10^{-12}$
16	Deleted	0	0	2	1	$0{\cdot}984465\times10^{-13}$
17	15	0	0	1	2	$0{\cdot}295264\times10^{-10}$
18	16	0	0	0	3	$0{\cdot}787643\times10^{-8}$

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