Module 2-3 Review of Probability Theory

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Outline

- Random variable
  - Probability distribution function
  - Survival function
  - Hazard function
  - Exponential distribution function
- Stochastic processes
- Markov process
  - Transition probability
  - Transition rate matrix

Stochastic Process

A collection of random variables.

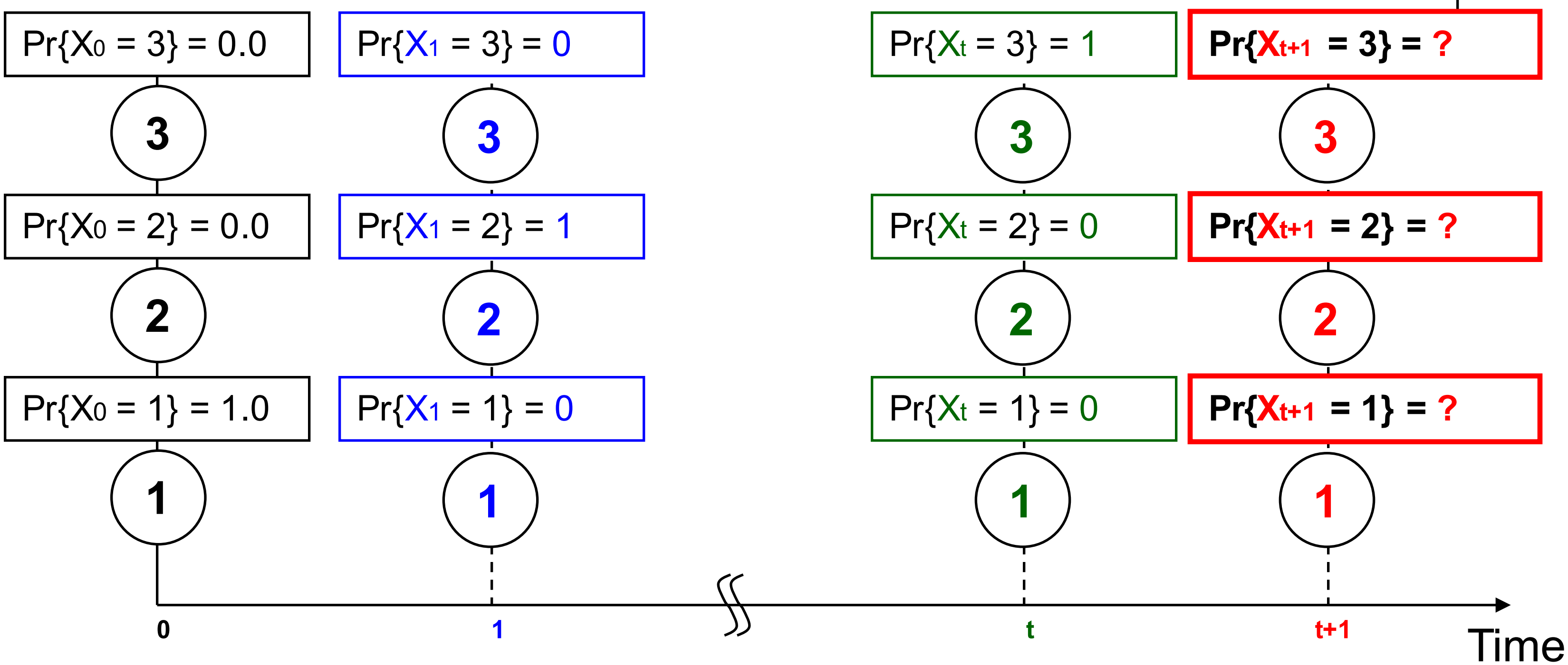
$$\{X_t, t \in T\}$$

- Discrete time process, t is discrete.
- Continuous time process, t is continuous.

Types of Stochastic Process

Continuous RV Continuous time process	Discrete RV Continuous time process
Continuous RV Discrete time process	Discrete RV Discrete time process

Discrete RV Discrete Time Process



Primary interest is to determine probability distribution of X_{t+1}.

Discrete RV Discrete Time Process

- If

$$P(X_n = x | X_m = y, X_l = z, \ldots) = P(X_n = x)$$

The stochastic process is said to be independent

- If

$$P(X_n = x | X_m = y, X_l = z, \ldots) = P(X_n = x | X_m = y)$$

The process is Markov process

The Chapman-Kolmogorov Equation

$$P(X_n = x | X_l = z) = \int_{-\infty}^{+\infty} P(X_n = x | X_m = y) P(X_m = y | X_l = z) dy$$

- This is for continuous state space and discrete time case

- For discrete states and discrete time, one step transition probability is defined as

$$P(X_n = x | X_{n-1} = y)$$

- If this transition probability is independent of n

$$P(X_n = x | X_{n-1} = y) = P(X_m = x | X_{m-1} = y)$$

- The process is time homogenous and transition probability is stationary

- The one step transition probability from state i to j is denoted by p_{ij}

$$\sum_{j} p_{ij} = 1$$

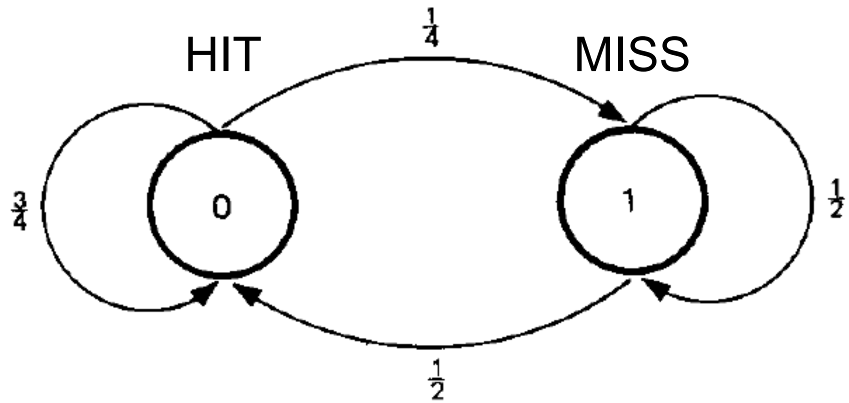
- It can be represented by a transition probability matrix P whose ijth element is the probability of transition from state i to j in one step

Example

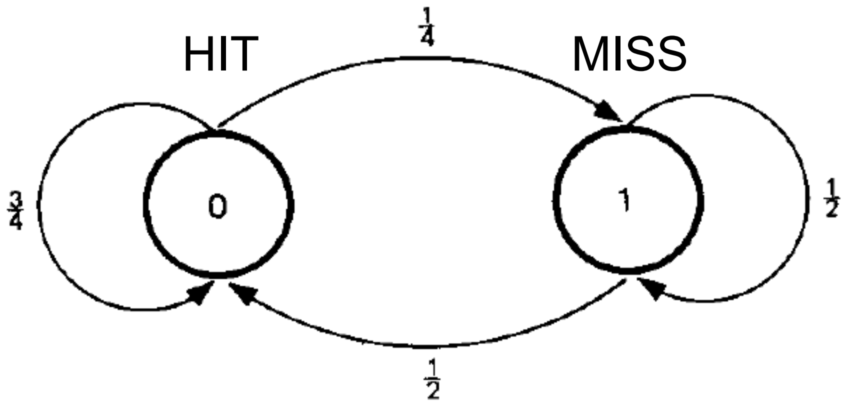
A person is practicing firing. If he misses, he becomes nervous and the probability of the next shot being a hit reduces to ½, but a hit bolsters his confidence and the chance of the next shot being a hit increases ¾.

(1) If the initial shot is a hit, what is the probability of a hit on the fourth shot?

(2) If the initial shot is a miss, what is the probability of a hit on the fourth shot?



Example



$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad P^3 = \begin{bmatrix} \frac{43}{64} & \frac{21}{64} \\ \frac{21}{32} & \frac{11}{32} \end{bmatrix}$$

(1) If the first shot being a hit

$$p^{(3)} = p^{(0)}P^3 = [1 \quad 0] \begin{bmatrix} \frac{43}{64} & \frac{21}{64} \\ \frac{21}{32} & \frac{11}{32} \end{bmatrix} = \begin{bmatrix} \frac{43}{64} & \frac{21}{64} \end{bmatrix}$$

(2) If the first shot being a miss

$$p^{(3)} = p^{(0)}P^3 = [0 \quad 1] \begin{bmatrix} \frac{43}{64} & \frac{21}{64} \\ \frac{21}{32} & \frac{11}{32} \end{bmatrix} = \begin{bmatrix} \frac{21}{32} & \frac{11}{32} \end{bmatrix}$$

Equilibrium Distribution

In the firing practice example

$$P^3 = \begin{bmatrix} 0.6719 & 0.3821 \\ 0.6563 & 0.3438 \end{bmatrix}$$

$$P^6 = \begin{bmatrix} 0.6667 & 0.3333 \\ 0.6665 & 0.3335 \end{bmatrix}$$

$$P^{12} = \begin{bmatrix} 0.6666 & 0.3334 \\ 0.6666 & 0.3334 \end{bmatrix}$$

Equilibrium Distribution

- In any Markov chain which is not cyclic the limit

$$x_j = \lim_{n\to\infty} p_j^{(n)}$$

- In any a-periodic irreducible Markov chain the above limit does not depend on the initial probability distribution so that

$$x_j = \lim_{n\to\infty} p_j^{(n)} = \lim_{n\to\infty} p_{ij}^{(n)}$$

- In a finite regular Markov chain, each row approaches a stationary probability vector $\alpha = (\alpha_0, \alpha_1, \dots)$
- This is called the unique stationary probability vector of the process and

$$\alpha P = \alpha$$

Equilibrium Distribution

In firing practice example

$$[\alpha_0 \quad \alpha_1]\begin{bmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} = [\alpha_0 \quad \alpha_1]$$

These two equations are identical, there fore and equation of the following form can be used

$$\alpha_0 + \alpha_1 = 1$$

Then

$$\alpha_0 = \frac{2}{3}, \alpha_1 = \frac{1}{3}$$

It can be seen that these values could also be obtained by multiplying P a large number of times

First Passage Time

- One parameter of interest in many Markov Chain problem is the time to encounter a state for the first time. This is called the **first passage time**.

- If this state is an absorbing state or has been made an absorbing state, this is called the time of absorption.

- In reliability engineering this concept is used to calculate the mean time to first failure, MTTFF.

First Passage Time

- It is possible to calculate mean and variance of first passage time by making state j and absorbing state and applying the ehrory of absorbing chains.

- An absorbing state is one which once entered cannot be left. The behavior of the stochastic process before once hitting state j will be the same as that of the original process.

- The first passage time from state i to state j is now the time of absorption starting from state i in the new process

First Passage Time

- The basic results for absorbing chain can be obtained from the fundamental matrix N

$$N = [I - Q]^{-1}$$

Where

N = the fundamental matrix whose n_{jk} denotes the mean number of times the process is in state k before absorption, the process having been started in state i

Q = The matrix obtained by deleting the jth row and the jth column from matrix P of transition probabilities

First Passage Time

$$N = [I - Q]^{-1}$$

- The mean first passage time from state i to j is therefore

$$\bar{t_i} = \sum_{k=1}^{N-1} n_{ik}$$

- The variance column vector is given by

$$W = [2N - I]\bar{t} - \bar{t_s}$$

where

W_i = The variance of the first passage time from state i to state j

$\bar{t}$ = The column vector such that $\bar{t_i}$ is the mean first passage time from I to j

$\bar{t_s}$ = The column vector with $\overline{t_{si} = \bar{t_i}^2}$